1. RATIO AND PROPORTION

A ratio is a comparison between two quantities. In making this comparison, both quantities must be expressed in terms of the same units.

Example:

Express the ratio of 1 hour to 1 day.

Solution:

A day contains 24 hours. The ratio is $\frac{1}{24}$, which can also be written 1 : 24.

Example:

Find the ratio of the shaded portion to the unshaded portion.

Solution:

There are 5 squares shaded out of 9. The ratio of the shaded portion to unshaded portion is $\frac{5}{4}$.

A proportion is a statement of equality between two ratios. The denominator of the first fraction and the numerator of the second are called the means of the proportion. The numerator of the first fraction and the denominator of the second are called the extremes. In solving a proportion, we use the theorem that states the product of the means is equal to the product of the extremes. We refer to this as *cross multiplying*.

Example:

Solve for *x:* $\frac{x+3}{5}=\frac{8-x}{6}$

Solution:

Cross multiply. $6x+18=40-5x$

$11x=22$

$x=2$

Example:

Solve for *x:* $4 : x = 9 : 18$

Solution:

Rewrite in fraction form. $\frac{4}{x}=\frac{9}{18}$

Cross multiply. $9x=72$

$x=8$

If you observe that the second fraction is equal to $\frac{1}{2}$, then the first must also be equal to $\frac{1}{2}$. Therefore, the missing denominator must be 8. Observation often saves valuable time.

Exercise 1

Work out each problem. Circle the letter that appears before your answer.

1. Find the ratio of 1 ft. 4 in. to 1 yd.
   (A) 1 : 3
   (B) 2 : 9
   (C) 4 : 9
   (D) 3 : 5
   (E) 5 : 12

2. A team won 25 games in a 40 game season. Find the ratio of games won to games lost.
   (A) $\frac{5}{8}$
   (B) $\frac{3}{8}$
   (C) $\frac{3}{5}$
   (D) $\frac{5}{3}$
   (E) $\frac{3}{2}$

3. In the proportion $a : b = c : d$, solve for d in terms of a, b and c.
   (A) $\frac{ac}{b}$
   (B) $\frac{bc}{a}$
   (C) $\frac{ab}{c}$
   (D) $\frac{a}{bc}$
   (E) $\frac{bc}{d}$

4. Solve for x: $\frac{x+1}{8} = \frac{28}{32}$
   (A) $6\frac{1}{2}$
   (B) 5
   (C) 4
   (D) 7
   (E) 6

5. Solve for y: $\frac{2y}{9} = \frac{y-1}{3}$
   (A) 3
   (B) $\frac{1}{3}$
   (C) $\frac{9}{15}$
   (D) $\frac{9}{4}$
   (E) $\frac{4}{9}$

2. DIRECT VARIATION

Two quantities are said to vary directly if they change in the same direction. As the first increases, the second does also. As the first decreases, the second does also.

For example, the distance you travel at a constant rate varies directly as the time spent traveling. The number of pounds of apples you buy varies directly as the amount of money you spend. The number of pounds of butter you use in a cookie recipe varies directly as the number of cups of sugar you use.

Whenever two quantities vary directly, a problem can be solved using a proportion. We must be very careful to compare quantities in the same order and in terms of the same units in both fractions. If we compare miles with hours in the first fraction, we must compare miles with hours in the second fraction.

You must always be sure that as one quantity increases or decreases, the other changes in the same direction before you try to solve using a proportion.

Example:

If 4 bottles of milk cost \$2, how many bottles of milk can you buy for \$8?

Solution:

The more milk you buy, the more it will cost. This is *direct*. We are comparing the number of bottles with cost.

$$\frac{4}{2} = \frac{x}{8}$$

If we cross multiply, we get $2x = 32$ or $x = 16$.

A shortcut in the above example would be to observe what change takes place in the denominator and apply the same change to the numerator. The denominator of the left fraction was multiplied by 4 to give the denominator of the right fraction. Therefore we multiply the numerator by 4 as well to maintain the equality. This method often means a proportion can be solved at sight with no written computation at all, saving valuable time.

Example:

If *b* boys can deliver *n* newspapers in one hour, how many newspapers can *c* boys deliver in the same time?

Solution:

The more boys, the more papers will be delivered. This is *direct*. We are comparing the number of boys with the number of newspapers.

$$\frac{b}{n} = \frac{c}{x}$$ Cross multiply and solve for x.

$$bx = cn$$

$$x = \frac{cn}{b}$$

Exercise 2

Work out each problem. Circle the letter that appears before your answer.

1. Find the cost, in cents, of 8 books if 3 books of the same kind cost *D* dollars.
   (A) $\frac{8D}{3}$
   (B) $\frac{3}{800D}$
   (C) $\frac{3}{8D}$
   (D) $\frac{800D}{3}$
   (E) $\frac{108D}{3}$

2. On a map $\frac{1}{2}$ inch = 10 miles. How many miles apart are two towns that are $2\frac{1}{4}$ inches apart on the map?
   (A) $11\frac{1}{4}$
   (B) 45
   (C) $22\frac{1}{2}$
   (D) $40\frac{1}{2}$
   (E) 42

3. The toll on the Intercoastal Thruway is 8¢ for every 5 miles traveled. What is the toll for a trip of 115 miles on this road?
   (A) $9.20
   (B) $1.70
   (C) $1.84
   (D) $1.64
   (E) $1.76

4. Mark's car uses 20 gallons of gas to drive 425 miles. At this rate, approximately how many gallons of gas will he need for a trip of 1000 miles?
   (A) 44
   (B) 45
   (C) 46
   (D) 47
   (E) 49

5. If *r* planes can carry *p* passengers, how many planes are needed to carry *m* passengers?
   (A) $\frac{rm}{p}$
   (B) $\frac{rp}{m}$
   (C) $\frac{p}{rm}$
   (D) $\frac{pm}{r}$
   (E) $\frac{m}{rp}$

3. INVERSE VARIATION

Two quantities are said to vary inversely if they change in opposite directions. As the first increases, the second decreases. As the first decreases, the second increases.

Whenever two quantities vary inversely, their product remains constant. Instead of dividing one quantity by the other and setting their quotients equal as we did in direct variation, we multiply one quantity by the other and set the products equal.

There are several situations that are good examples of inverse variation.

A) The number of teeth in a meshed gear varies inversely as the number of revolutions it makes per minute. The more teeth a gear has, the fewer revolutions it will make per minute. The less teeth it has, the more revolutions it will make per minute. The product of the number of teeth and the revolutions per minute remains constant.

B) The distance a weight is placed from the fulcrum of a balanced lever varies inversely as its weight. The heavier the object, the shorter must be its distance from the fulcrum. The lighter the object, the greater must be the distance. The product of the weight of the object and its distance from the fulcrum remains constant.

C) When two pulleys are connected by a belt, the diameter of a pulley varies inversely as the number of revolutions per minute. The larger the diameter, the smaller the number of revolutions per minute. The smaller the diameter, the greater the number of revolutions per minute. The product of the diameter of a pulley and the number of revolutions per minute remains constant.

D) The number of people hired to work on a job varies inversely as the time needed to complete the job. The more people working, the less time it will take. The fewer people working, the longer it will take. The product of the number of people and the time worked remains constant.

E) How long food, or any commodity, lasts varies inversely as the number of people who consume it. The more people, the less time it will last. The fewer people, the longer it will last. The product of the number of people and the time it will last remains constant.

Example:

If 3 men can paint a house in 2 days, how long will it take 2 men to do the same job?

Solution:

The fewer men, the more days. This is *inverse*.

$$3 \cdot 2 = 2 \cdot x$$
$$6 = 2x$$
$$x = 3 \text{ days}$$

Exercise 3

Work out each problem. Circle the letter that appears before your answer.

1. A field can be plowed by 8 machines in 6 hours. If 3 machines are broken and cannot be used, how many hours will it take to plow the field?
   (A) 12
   (B) $9\frac{3}{5}$
   (C) $3\frac{3}{4}$
   (D) 4
   (E) 16

2. Camp Starlight has enough milk to feed 90 children for 4 days. If 10 of the children do not drink milk, how many days will the supply last?
   (A) 5
   (B) 6
   (C) $4\frac{1}{2}$
   (D) $4\frac{1}{8}$
   (E) $5\frac{1}{3}$

3. A pulley revolving at 200 revolutions per minute has a diameter of 15 inches. It is belted to a second pulley which revolves at 150 revolutions per minute. Find the diameter, in inches, of the second pulley.
   (A) 11.2
   (B) 20
   (C) 18
   (D) 16.4
   (E) 2

4. Two boys weighing 60 pounds and 80 pounds balance a seesaw. How many feet from the fulcrum must the heavier boy sit if the lighter boy is 8 feet from the fulcrum?
   (A) 10
   (B) $10\frac{2}{3}$
   (C) 9
   (D) $7\frac{1}{2}$
   (E) 6

5. A gear with 20 teeth revolving at 200 revolutions per minute is meshed with a second gear turning at 250 revolutions per minute. How many teeth does this gear have?
   (A) 16
   (B) 25
   (C) 15
   (D) 10
   (E) 24

In solving variation problems, you must decide whether the two quantities involved change in the same direction, in which case it is direct variation and should be solved by means of proportions. If the quantities change in opposite directions, it is inverse variation, solved by means of constant products. In the following exercises, decide carefully whether each is an example of direct or inverse variation.

Exercise 4

Work out each problem. Circle the letter that appears before your answer.

1. A farmer has enough chicken feed to last 30 chickens for 4 days. If 10 more chickens are added, how many days will the feed last?

   (A) 3
   (B) $1\frac{1}{3}$
   (C) 12
   (D) $2\frac{2}{3}$
   (E) $5\frac{1}{3}$

2. At c cents per can, what is the cost of p cases of soda if there are 12 cans in a case?

   (A) $12cp$
   (B) $\frac{cp}{12}$
   (C) $\frac{12}{cp}$
   (D) $\frac{12p}{c}$
   (E) $\frac{12c}{p}$

3. If m boys can put up a fence in d days, how many days will it take to put up the fence if two of the boys cannot participate?

   (A) $\frac{d}{-2}$
   (B) $\frac{d(m-2)}{m}$
   (C) $\frac{md}{m-2}$
   (D) $\frac{m-2}{md}$
   (E) $\frac{m(m-2)}{d}$

4. A recipe calls for $\frac{3}{4}$ lb. of butter and 18 oz. of sugar. If only 10 oz. of butter are available, how many ounces of sugar should be used?

   (A) $13\frac{1}{2}$
   (B) 23
   (C) 24
   (D) 14
   (E) 15

5. If 3 kilometers are equal to 1.8 miles, how many kilometers are equal to 100 miles?

   (A) 60
   (B) $166\frac{2}{3}$
   (C) 540
   (D) $150\frac{1}{2}$
   (E) 160.4

RETEST

Work out each problem. Circle the letter that appears before your answer.

1. Solve for *x*: $\frac{3x}{8} = \frac{x+7}{12}$
   (A) $\frac{7}{28}$
   (B) 2
   (C) 4
   (D) $2\frac{3}{4}$
   (E) 1

2. Solve for *x* if $a = 5$, $b = 8$, and $c = 3$: $\frac{a-3}{x} = \frac{b+2}{5c}$
   (A) 5
   (B) 20
   (C) 2
   (D) 3
   (E) 6

3. A map is drawn to a scale of $\frac{1}{2}$ inch = 20 miles. How many miles apart are two cities that are $3\frac{1}{4}$ inches apart on the map?
   (A) 70
   (B) 130
   (C) 65
   (D) $32\frac{1}{2}$
   (E) 35

4. Mr. Weiss earned $12,000 during the first 5 months of the year. If his salary continues at the same rate, what will his annual income be that year?
   (A) $60,000
   (B) $28,000
   (C) $27,000
   (D) $30,000
   (E) $28,800

5. How many pencils can be bought for *D* dollars if *n* pencils cost *c* cents?
   (A) $\frac{nD}{c}$
   (B) $\frac{nD}{100c}$
   (C) $\frac{100D}{nc}$
   (D) $\frac{100nD}{c}$
   (E) $\frac{nc}{100D}$

6. Ten boys agree to paint the gym in 5 days. If five more boys join in before the work begins, how many days should the painting take?
   (A) $3\frac{1}{3}$
   (B) $3\frac{1}{2}$
   (C) 10
   (D) $2\frac{1}{2}$
   (E) $2\frac{3}{4}$

7. A weight of 120 pounds is placed five feet from the fulcrum of a lever. How far from the fulcrum should a 100 pound weight be placed in order to balance the lever?
   (A) 6 ft.
   (B) $4\frac{1}{6}$ ft.
   (C) $5\frac{1}{2}$ ft.
   (D) $6\frac{1}{2}$ ft.
   (E) $6\frac{2}{3}$ ft.

8. A photograph negative measures $1\frac{7}{8}$ inches by $2\frac{1}{2}$ inches. The printed picture is to have its longer dimension be 4 inches. How long should the shorter dimension be?
   (A) $2\frac{3}{8}$"
   (B) $2\frac{1}{2}$"
   (C) 3″
   (D) $3\frac{1}{8}$"
   (E) $3\frac{3}{8}$"

9. A gear with 60 teeth is meshed to a gear with 40 teeth. If the larger gear revolves at 20 revolutions per minute, how many revolutions does the smaller gear make in a minute?
(A) $13\frac{1}{3}$
(B) 3
(C) 300
(D) 120
(E) 30

10. How many gallons of paint must be purchased to paint a room containing 820 square feet of wall space, if one gallon covers 150 square feet? (Any fraction must be rounded *up*.)
(A) 4
(B) 5
(C) 6
(D) 7
(E) 8

SOLUTIONS TO PRACTICE EXERCISES

Diagnostic Test

1. (B)
$$2x(4) = 3(x + 5)$$
$$8x = 3x + 15$$
$$5x = 15$$
$$x = 3$$

2. (E)
$$\frac{4}{x} = \frac{10}{20} \quad \text{Cross multiply.}$$
$$80 = 10x$$
$$x = 8$$

3. (C) We compare inches to miles.
$$\frac{2}{25} = \frac{5\frac{2}{5}}{x} \quad \text{Cross multiply.}$$
$$2x = 135$$
$$x = 67\frac{1}{2}$$

4. (A) We compare apples to cents.
$$\frac{x}{c} = \frac{n}{d}$$
$$dx = nc \quad \text{Cross multiply.}$$
$$x = \frac{nc}{d}$$

5. (B) We compare miles to months.
$$\frac{5}{7000} = \frac{12}{x}$$
$$5x = 84{,}000$$
$$x = 16{,}800$$

6. (E) Number of teeth times speed remains constant.
$$20 \cdot 30 = x \cdot 25$$
$$600 = 25x$$
$$x = 24$$

7. (B) Weight times distance from the fulcrum remains constant.
$$90 \cdot 3 = 50 \cdot x$$
$$270 = 50x$$
$$x = 5\frac{2}{5} \text{ ft.}$$

8. (C) The more dogs, the fewer days. This is inverse variation.
$$2 \cdot 3 = 3 \cdot x$$
$$6 = 3x$$
$$x = 2 \text{ weeks} = 14 \text{ days}$$

9. (D) Number of machines times hours needed remains constant.
$$m \cdot h = (m - 2) \cdot x$$
$$x = \frac{mh}{m - 2}$$

10. (C) The more men, the fewer days. This is inverse variation.
$$20 \cdot 6 = 24 \cdot x$$
$$120 = 24x$$
$$x = 5$$

The rations will last 1 day less.

Exercise 1

1. (C) 1 ft. 4 in. = 16 in.

   1 yd. = 36 in.

   $$\frac{16}{36} = \frac{4}{9}$$

2. (D) The team won 25 games and lost 15.

   $$\frac{25}{15} = \frac{5}{3}$$

3. (B) $\frac{a}{b} = \frac{c}{d}$ Cross multiply. Divide by *a*.

   $$ad = bc$$

   $$d = \frac{bc}{a}$$

4. (E) $32(x + 1) = 28(8)$

   $$32x + 32 = 224$$

   $$32x = 192$$

   $$x = 6$$

5. (A) $9(y - 1) = 2y(3)$

   $$9y - 9 = 6y$$

   $$3y = 9$$

   $$y = 3$$

Exercise 2

1. (D) We compare books with cents. *D* dollars is equivalent to 100*D* cents.

   $$\frac{3}{100D} = \frac{8}{x}$$

   $$3x = 800D$$

   $$x = \frac{800D}{3}$$

2. (B) We compare inches to miles.

   $$\frac{\frac{1}{2}}{10} = \frac{2\frac{1}{4}}{x}$$

   $\frac{1}{2}x = 22\frac{1}{2}$ Cross multiply. Multiply by 2.

   $$x = 45$$

3. (C) We compare cents to miles.

   $$\frac{8}{5} = \frac{x}{115}$$

   $5x = 920$ Cross multiply.

   $$x = \$1.84$$

4. (D) We compare gallons to miles.

   $$\frac{20}{425} = \frac{x}{1000}$$

   $425x = 20,000$ Cross multiply. To avoid large numbers, divide by 25.

   $$17x = 800$$

   $$x = 47\frac{1}{17}$$

5. (A) We compare planes to passengers.

   $\frac{r}{p} = \frac{x}{m}$ Cross multiply. Divide by *p*.

   $$px = rm$$

   $$x = \frac{rm}{p}$$

Exercise 3

1. (B) Number of machines times hours needed remains constant.

$$8 \cdot 6 = 5 \cdot x$$
$$48 = 5x$$
$$x = 9\frac{3}{5}$$

2. (C) Number of children times days remains constant.

$$90 \cdot 4 = 80 \cdot x$$
$$80x = 360$$
$$x = 4\frac{1}{2}$$

3. (B) Diameter times speed remains constant.

$$15 \cdot 200 = x \cdot 150$$
$$3000 = 150x$$
$$x = 20$$

4. (E) Weight times distance from fulcrum remains constant.

$$80 \cdot x = 60 \cdot 8$$
$$80x = 480$$
$$x = 6$$

5. (A) Number of teeth times speed remains constant.

$$20 \cdot 200 = x \cdot 250$$
$$250x = 4000$$
$$x = 16$$

Exercise 4

1. (A) The more chickens, the fewer days. This is *inverse*.

$$30 \cdot 4 = 40 \cdot x$$
$$40x = 120$$
$$x = 3$$

2. (A) The more cases, the more cents. This is *direct*. We compare cents with cans. In p cases there will be $12p$ cans.

$$\frac{c}{1} = \frac{x}{12p}$$
$$x = 12cp$$

3. (C) The fewer boys, the more days. This is *inverse*.

$$m \cdot d = (m-2) \cdot x$$
$$\frac{md}{m-2} = x$$

4. (E) The less butter, the less sugar. This is *direct*. Change $\frac{3}{4}$ lb. to 12 oz.

$$\frac{12}{18} = \frac{10}{x}$$
$$12x = 180$$
$$x = 15$$

5. (B) The more kilometers, the more miles. This is *direct*.

$$\frac{3}{1.8} = \frac{x}{100}$$
$$1.8x = 300$$
$$18x = 3000$$
$$x = 166\frac{2}{3}$$